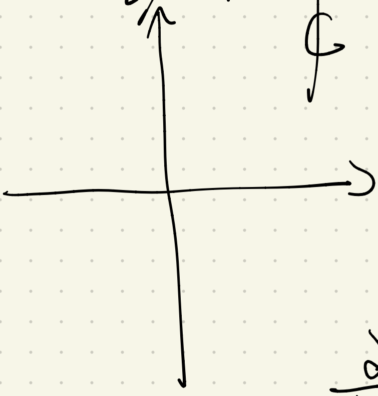
13/10 MATH 2230 A



$\mathbb{C}$ (x, y)

(r, θ)

$$x = \frac{z+\bar{z}}{2}$$

$$y = \frac{z-\bar{z}}{2i} \qquad f(z, \bar{z})$$

$$\frac{\partial}{\partial \bar{z}} f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}.$$

If f is analytic,

$(f = u + iv)$

then
$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}(u_x + i v_x) - \frac{1}{2i}(u_y + i v_y)$$

$$= \frac{1}{2}u_x + \frac{1}{2}v_x i + \frac{u_y}{2} i - \frac{1}{2}v_y$$

$$= \frac{1}{2}\underbrace{(u_x - v_y)}_{=0} + \frac{1}{2}\underbrace{(v_x + u_y)}_{=0} i$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \frac{\partial}{\partial \bar{z}} f = 0.$$

e.g. $f = x = \frac{z+\bar{z}}{2}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \neq 0.$$

1. Integral of a complex value function w.r.t $\mathbb{R}$ variable.

View i as a constant, and integrate as in the real case.

a) $\int_0^1 (1+\underbrace{it}_{\text{constant}})^2 \, dt$

$= \frac{1}{3i}(1+it)^3 \Big|_0^1$

$= \frac{1}{3i}\left((1+i)^3 - 1\right)$

$= \frac{2}{3} + i.$

2. Upper Bound for Moduli of Contour Integrals.

$$\left| \int_a^b w(t)\, dt \right| \leq \int_a^b |w(t)|\, dt .$$

Proof: Assume $\int_a^b w(t)\, dt = \underline{re^{i\theta}}$ Constant

, $\int_a^b \underbrace{w(t)\, e^{-i\theta}}_{\text{New } \mathbb{C}\text{-valued function.}}\, dt = r$. (*)

$$\int_a^b U + iV\, dt = \int_a^b U\, dt + \boxed{i \int_a^b V\, dt}$$

(*) implies that $\boxed{} = 0$.

So (*) can be rewritten as

$$\int_a^b \mathrm{Re}\left(w(t)\, e^{-i\theta}\right) dt = r$$

We notice that

$$\mathrm{Re}\left(w(t)\, e^{-i\theta}\right) \leq \left| w(t)\, e^{-i\theta} \right| = |w(t)| .$$

$\Rightarrow \quad r \leq \int_a^b |w(t)|\, dt$

$r = \left| \int_a^b w(t)\, dt \right| .$ $\square$.

For a contour, we just parametrise with

$\gamma : [0, 1] \to \mathbb{C}$.

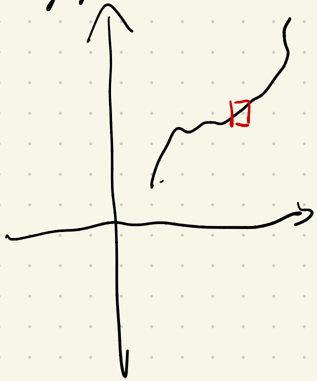
So $\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t))\, \gamma'(t)\, dt$

Apply the above inequality,

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_0^1 |f(\gamma(t))|\, |\gamma'(t)|\, dt .$$

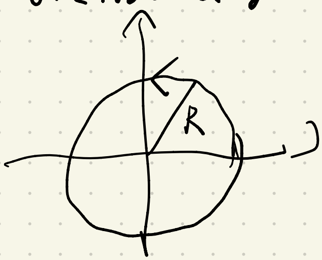
By Extreme Value Thm, and as γ is compact, $|f|$ achieves its maximum M on γ.

$$\left|\int_\gamma f(z)\,dz\right| \le M \underbrace{\boxed{\int_0^1 |\gamma'(t)|\,dt}}_{\text{Length of } \gamma.\ (2)} = ML.$$



$$|f(z)\,w| \le M|w|.$$

8. (P130 Q4)



1. $$\left|\int_{C_R} \frac{\operatorname{Log} z}{z^2}\,dz\right| \le \max_{z\in C_R} \frac{|\operatorname{Log} z|}{|z|^2}\cdot 2\pi R$$

$$= \max_{\substack{|z|=R\\ -\pi<\theta\le\pi}} \frac{|\ln R+\theta i|}{R^2}\cdot 2\pi R$$

$$= \left(\frac{\ln R+\pi}{R^2}\right)2\pi R = 2\pi\left(\frac{\ln R+\pi}{R}\right)$$

2. $$\lim_{R\to\infty} 2\pi\left(\frac{\ln R+\pi}{R}\right) = 2\pi \lim_{R\to\infty}\frac{\frac{1}{R}}{1} = 0.$$

3. Antiderivatives.

If f is continuous in D, then

i) f has an antiderivative F. e.g. $F'(z) = f(z)$.

ii) for $\forall \gamma \subset D$, the 2 end pts are z_1 & z_2.

$$\int_\gamma f(z) = F(z_2) - F(z_1).$$

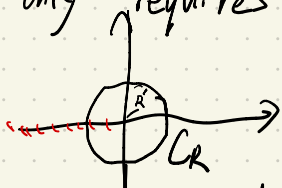
iii) If contour γ is closed, then $\int_\gamma f(z) = 0$.

i), ii), iii) are equivalent.

Remark: i) The 3 statements are equivalent, not claim any of them is true for a continuous function.

ii) It only requires continuity.

e.g. $\frac{1}{z}$



$R > 0$.

$$\int_{C_R} \frac{1}{z}\, dz = 0 \quad ? \quad \text{No.}$$

$z = Re^{i\theta}$, $\theta : -\pi \to \pi$.

$$\int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}}\, iRe^{i\theta}\, d\theta = 2\pi i. \quad \text{(Residue Theorem).}$$

e.g. 10(P147. Q7)

b) $\int_0^{\pi+2i} \cos\frac{z}{2}\,dz$, $f = \cos\frac{z}{2}$, $F = 2\sin\frac{z}{2}$

$= 2\sin\frac{z}{2}\Big|_0^{\pi+2i}$

$= 2\left(\sin\frac{\pi+2i}{2} - 0\right)$

sin

cos

$= 2\sin\frac{\pi+2i}{2}$

$= 2\,\dfrac{e^{i\frac{\pi+2i}{2}} - e^{-i\frac{\pi+2i}{2}}}{2i}$

$= \dfrac{e^{\frac{\pi}{2}i}e^{-1} - e^{-\frac{\pi}{2}i+1}}{i}$

$= \dfrac{ie^{-1} + ie}{i}$

$= e^{-1} + e$. ✓ 100.